CS580:
Monte Carlo Integration

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KAIST

Deadlines

- Declare project team members
  - By 4/1 at KLMS
  - Confirm schedules of paper talks and project talks at 4/2

- Declare two papers for student presentations
  - by 4/10 at KLMS
  - Discuss them at the class of 4/11

Paper Presentation: Expectations

- Good summary, not full detail, of the paper
  - Talk about motivations of the work
  - Given proper background on the related work
  - Explain main idea and results of the paper
  - Discuss strengths and weaknesses of the method

- Prepare an overview slide
  - Talk about most important things and connect them well

High-Level Ideas

- Delivers most important ideas and results
  - Do not talk about minor details
  - Prepare some slides for important details, since they can be asked

- Spend most time to figure out the most important things and prepare good slides for them

- If possible, connect it to your main project

Be Honest

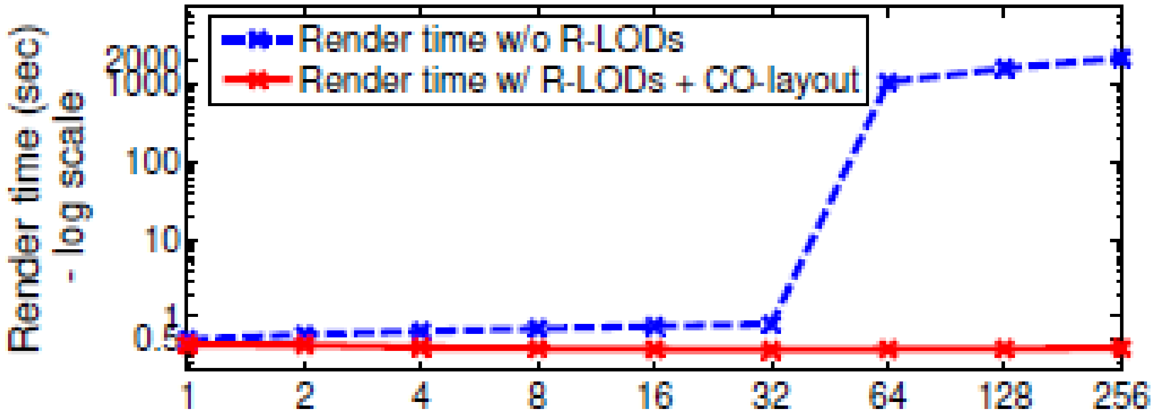
- Do not skip important ideas that you don’t know
  - Explain as much as you know and mention that you don’t understand some parts

- If you get questions you don’t know good answers, just say it

- In the end, you need to explain them before the semester ends

Result Presentation

- Give full experiment settings and present data with the related information



- After showing the data, give a message that we can pull of the data
- Show images/videos, if there are

Utilizing Existing Resources

- Use author’s slides, codes, and video, if they exist
- Give proper credits or citations
  - Without them, you are cheating!

Prepare Quiz

- Give two simple questions to draw attentions
  - Ask a keyword
  - Simple true or false questions
  - Multiple choice questions

- Grade them in the scale of 0 and 10, and send the score to TA

Audience feedback form

Date:
Talk title:
Speaker:

A. Was the talk well organized and well prepared?
5: Excellent 4: good 3: okay 2: less than average 1: poor

B. Was the talk comprehensible? How well were important concepts covered?
5: Excellent 4: good 3: okay 2: less than average 1: poor

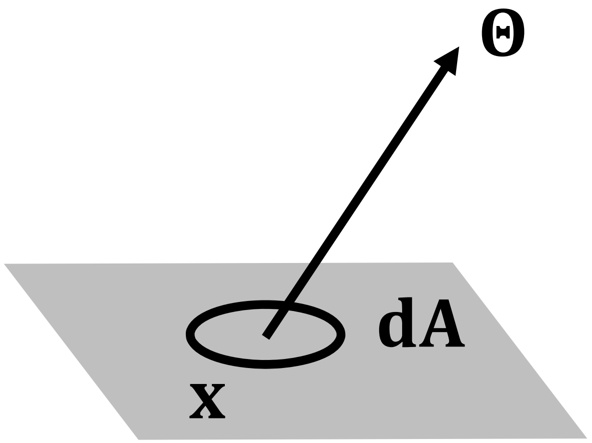
Any comments to the speaker

Class Objectives (Ch. 14)

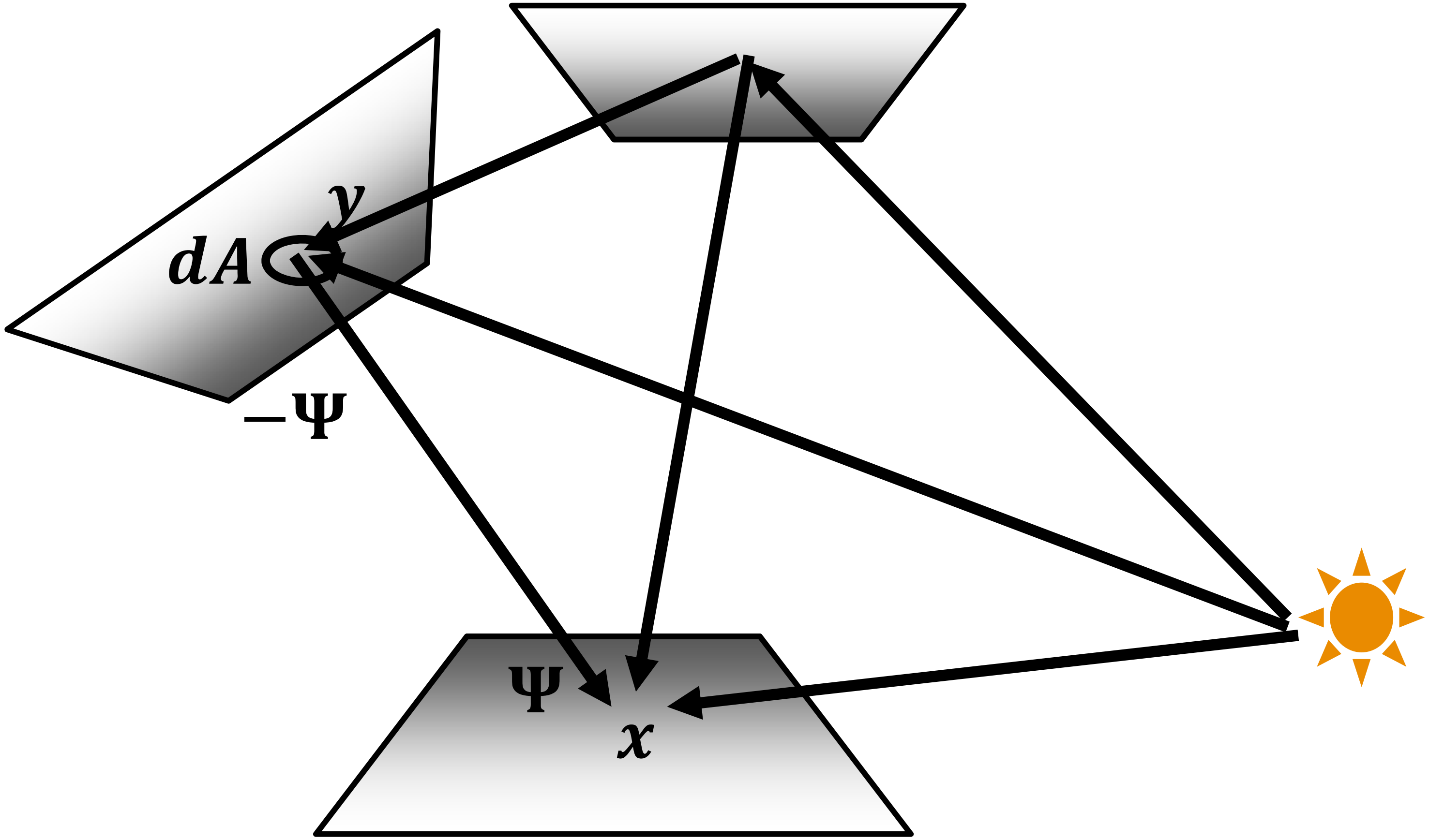
- **Sampling approach for solving the rendering equation**
  - **Monte Carlo integration**
  - **Estimator and its variance**

Radiance Evaluation

- Fundamental problem in GI algorithm
  - Evaluate radiance at a given surface point in a given direction
  - Invariance defines radiance everywhere else



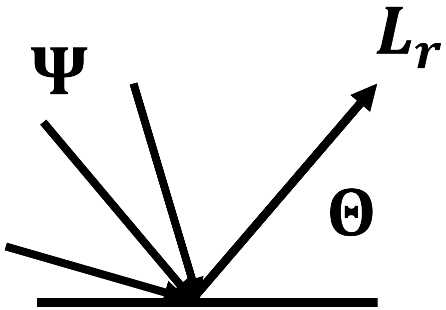
We need to find many paths…



Why Monte Carlo?

- Radiance is hard to evaluate

$$L_r(x \rightarrow \Theta) = \int_{\Psi} L(x \leftarrow \Psi) f_r(x, \Psi \rightarrow \Theta) \cos \theta_x dw_{\Psi},$$



- Sample many paths
  - Integrate over all incoming directions
- Analytical integration is difficult
  - Need numerical techniques

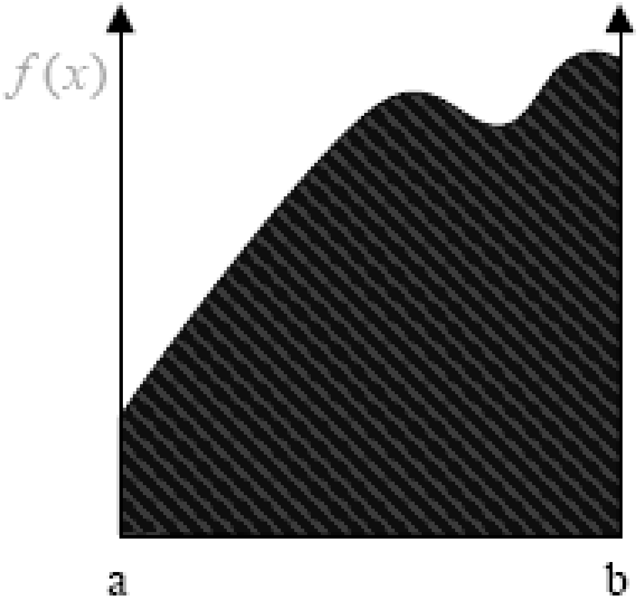
Monte Carlo Integration

- Numerical tool to evaluate integrals
  - Use sampling
- Stochastic errors
- Unbiased
  - On average, we get the right answer

Numerical Integration

- A one-dimensional integral:

$$I = \int_a^b f(x)dx$$

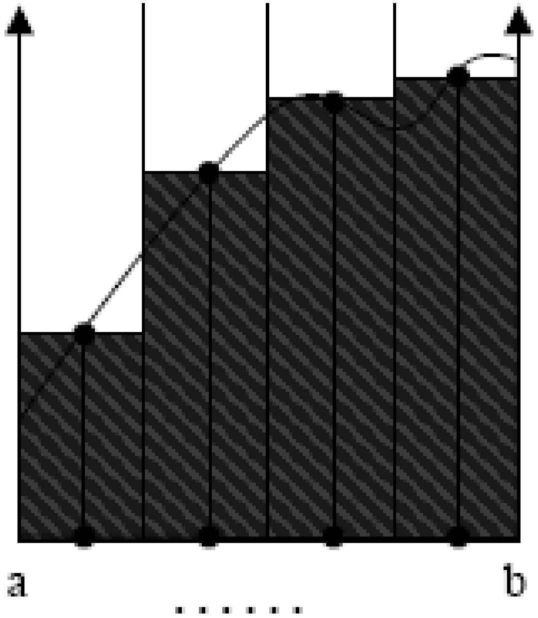


Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)dx$$

$$\approx \sum_{i=1}^{N} w_i f(x_i)$$

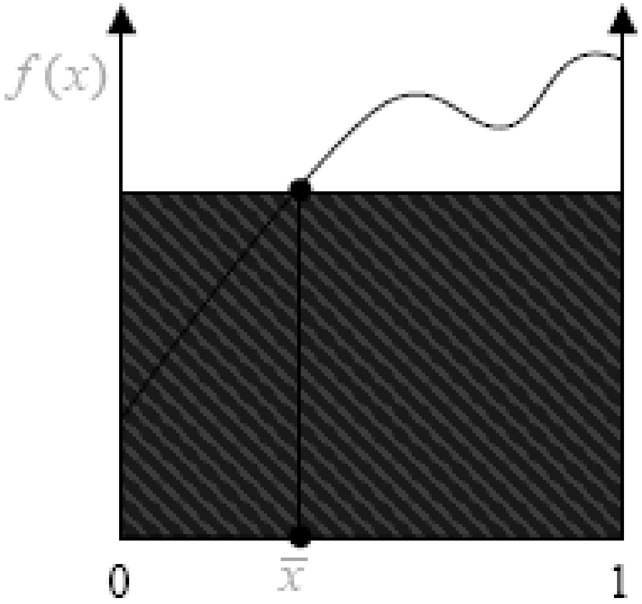


Monte Carlo Integration

Primary estimator:

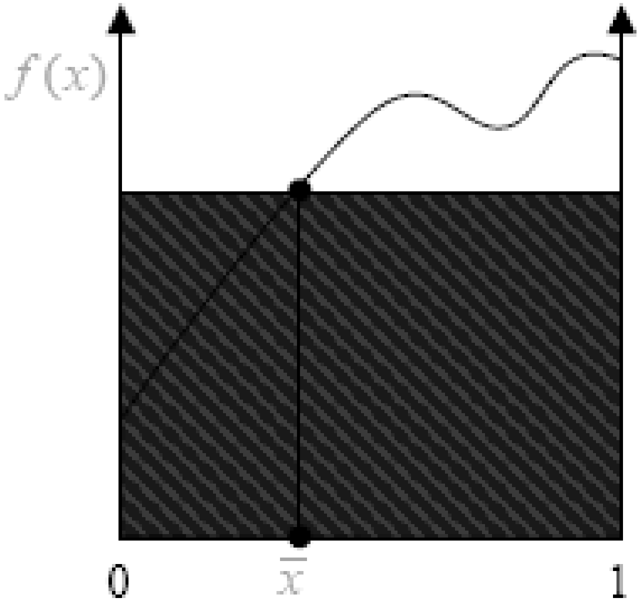
$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(\bar{x})$$



Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

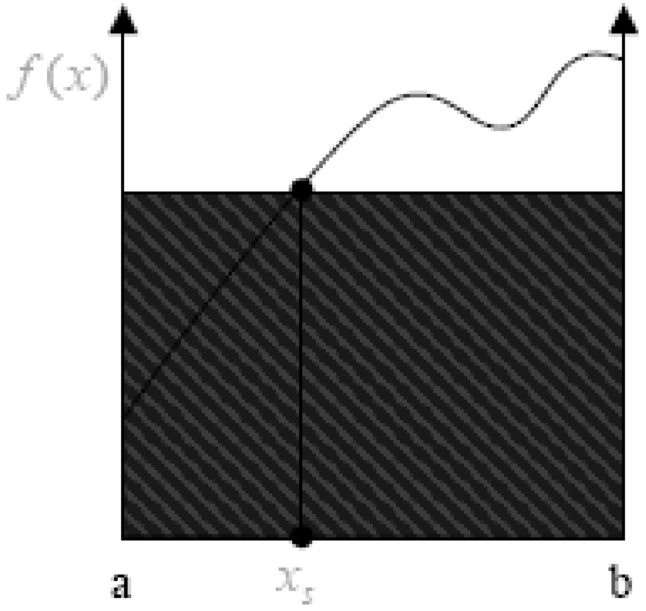
$$I_{prim} = f(\bar{x})$$

$$E(I_{prim}) = \int_0^1 f(x)p(x)dx = \int_0^1 f(x)1dx = I$$

Unbiased estimator!

Monte Carlo Integration

Primary estimator:



$$I = \int_a^b f(x)dx$$

$$I_{prim} = f(x_s)(b-a)$$

$$E(I_{prim}) = \int_a^b f(x)(b-a)p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!

Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

$$\sigma^2_{prim} = \int_a^b [\frac{f(x)}{p(x)} - I]^2 p(x)dx$$

- **Consider p(x) for estimate**
- **We will study it as importance sampling later**

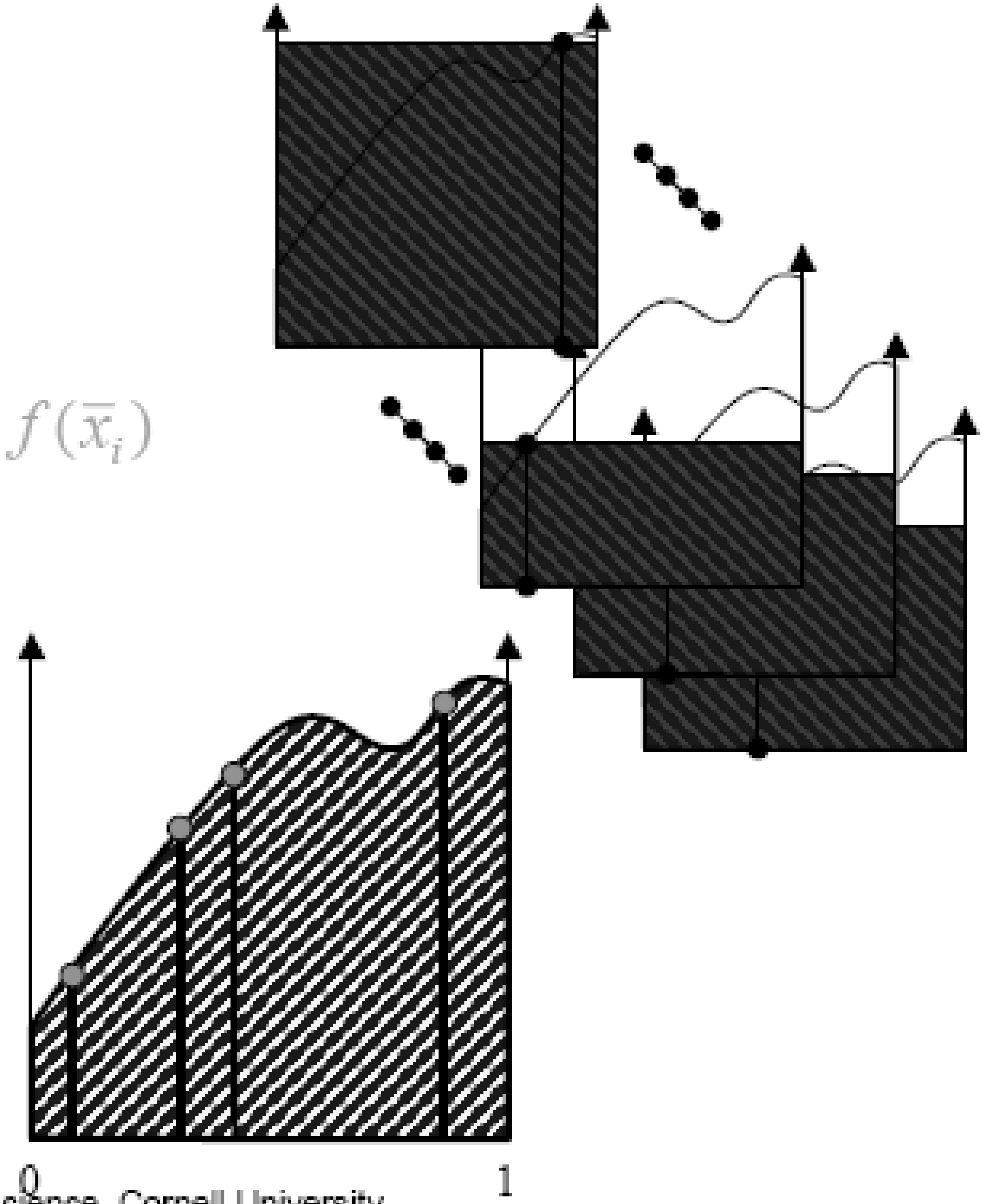
More samples

Secondary estimator

Generate N random samples x_i

Estimator: $\langle I \rangle = I_{\text{sec}} = \frac{1}{N}\sum_{i=1}^{N} f(\bar{x}_i)$

Variance $\sigma_{\text{sec}}^2 = \sigma_{prim}^2 / N$



Mean Square Error of MC Estimator

- **MSE**

$$MSE(\hat{Y}) = E[(\hat{Y} - Y)^2] = \frac{1}{N}\sum_i(\hat{Y}_i - Y_i)^2.$$

- **Decomposed into bias and variance terms**

$$\begin{aligned} MSE(\hat{Y}) &= E\left[(\hat{Y} - E[\hat{Y}])^2\right] + (E(\hat{Y}) - Y)^2 \\ &= Var(\hat{Y}) + Bias(\hat{Y}, Y)^2. \end{aligned}$$

- **Bias: how far the estimation is away from the ground truth**
- **Variance: how far the estimation is away from its average estimator**

Bias of MC Estimator

$$\begin{aligned}
E[\hat{I}] &= E\left[\frac{1}{N}\sum_i \frac{f(x_i)}{p(x_i)}\right] \\
&= \frac{1}{N}\int \sum_i \frac{f(x_i)}{p(x_i)} p(x)dx \\
&= \frac{1}{N}\sum_i \int \frac{f(x)}{p(x)} p(x)dx, \because x_i \text{ samples have the same } p(x) \\
&= \frac{N}{N}\int f(x)dx = I. \qquad (14.6)
\end{aligned}$$

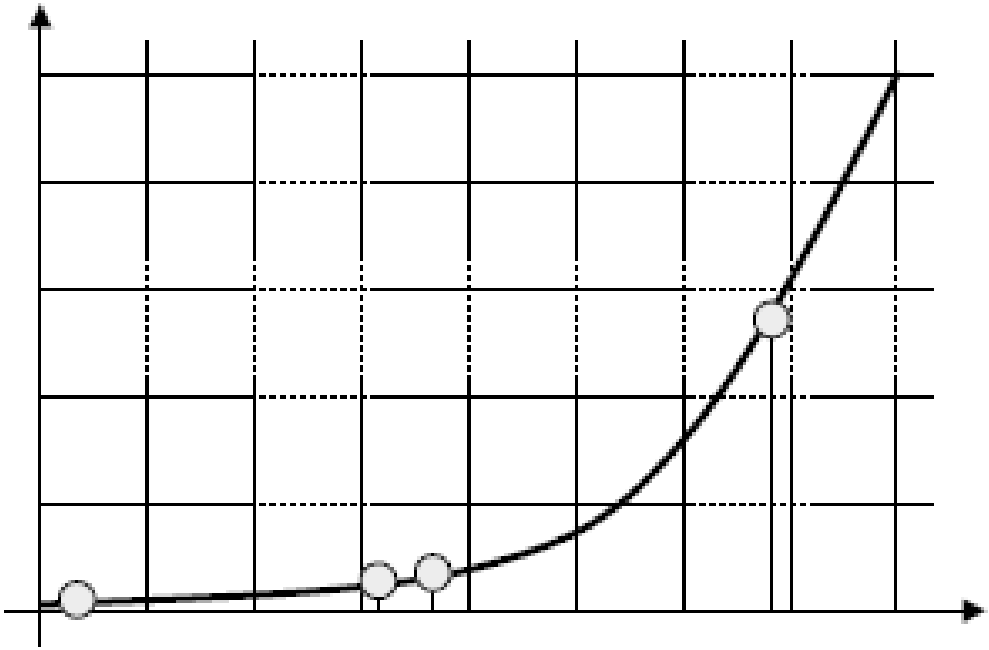
- **On average, it gives the right answer: unbiased**

Variance of MC Estimator

$$
\begin{aligned}
Var(\hat{I}) &= Var\Big(\frac{1}{N}\sum_i \frac{f(x_i)}{p(x_i)}\Big) \\
&= \frac{1}{N^2} Var\Big(\sum_i \frac{f(x_i)}{p(x_i)}\Big) \\
&= \frac{1}{N^2} \sum_i Var\Big(\frac{f(x_i)}{p(x_i)}\Big), \because x_i \text{ samples are independent from each other.} \\
&= \frac{1}{N^2} N Var\Big(\frac{f(x)}{p(x)}\Big), \because x_i \text{ samples are from the same distribution.} \\
&= \frac{1}{N} Var\Big(\frac{f(x)}{p(x)}\Big) = \frac{1}{N} \int \left(\frac{f(x)}{p(x)} - E\left[\frac{f(x)}{p(x)}\right]\right)^2 p(x)dx. \quad (14.7)
\end{aligned}
$$

MC Integration - Example

- Integral $I = \int_0^1 5x^4 dx = 1$
- Uniform sampling
- Samples :



$x_1 =$ .86 $\langle I \rangle = 2.74$

$x_2 =$ .41 $\langle I \rangle = 1.44$

$x_3 =$ .02 $\langle I \rangle = 0.96$

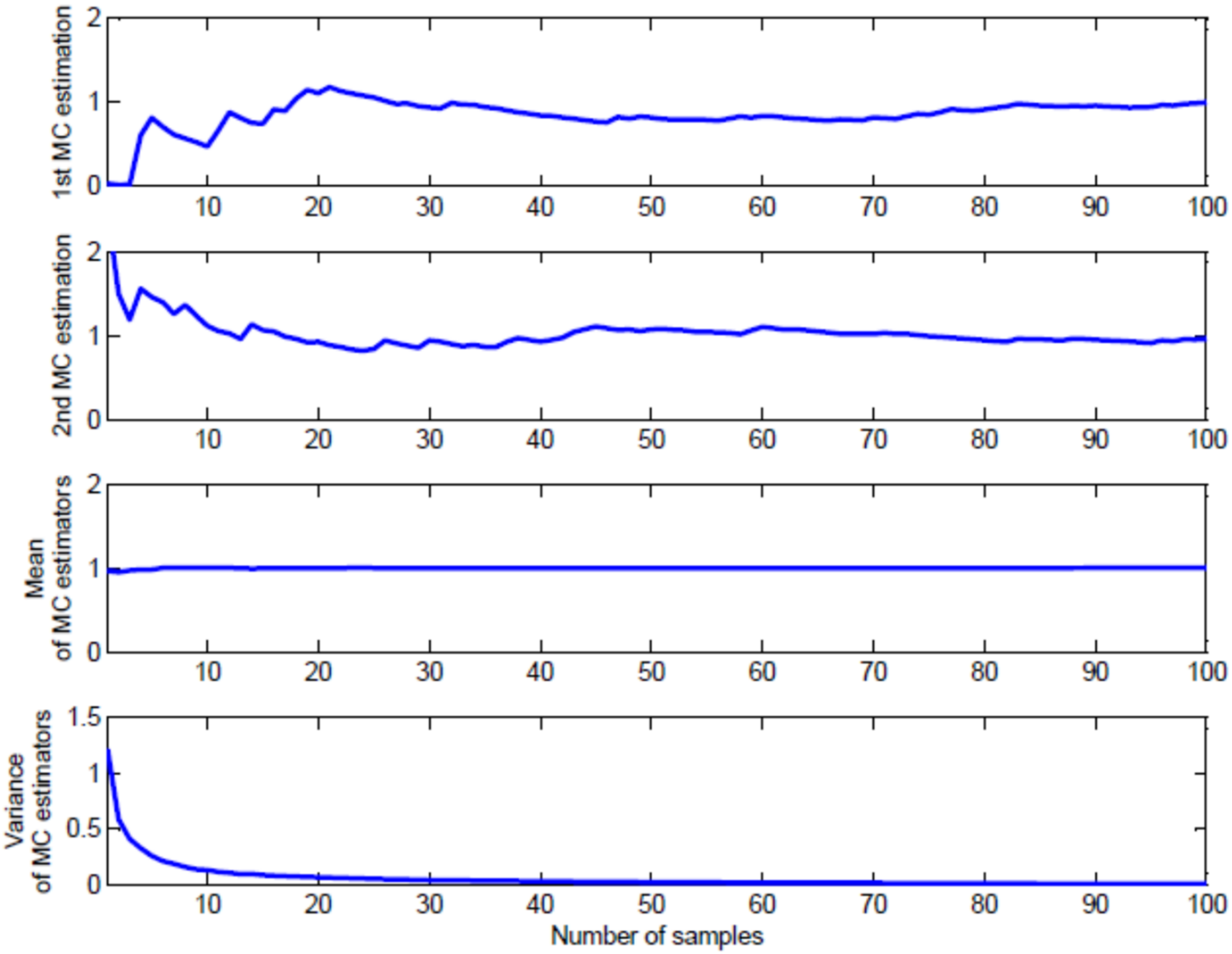
$x_4 =$ .38 $\langle I \rangle = 0.75$

MC Integration - Example

- **Integral**

$$I = \int_0^1 4x^3 dx = 1$$

$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} 4x_i^3,$$

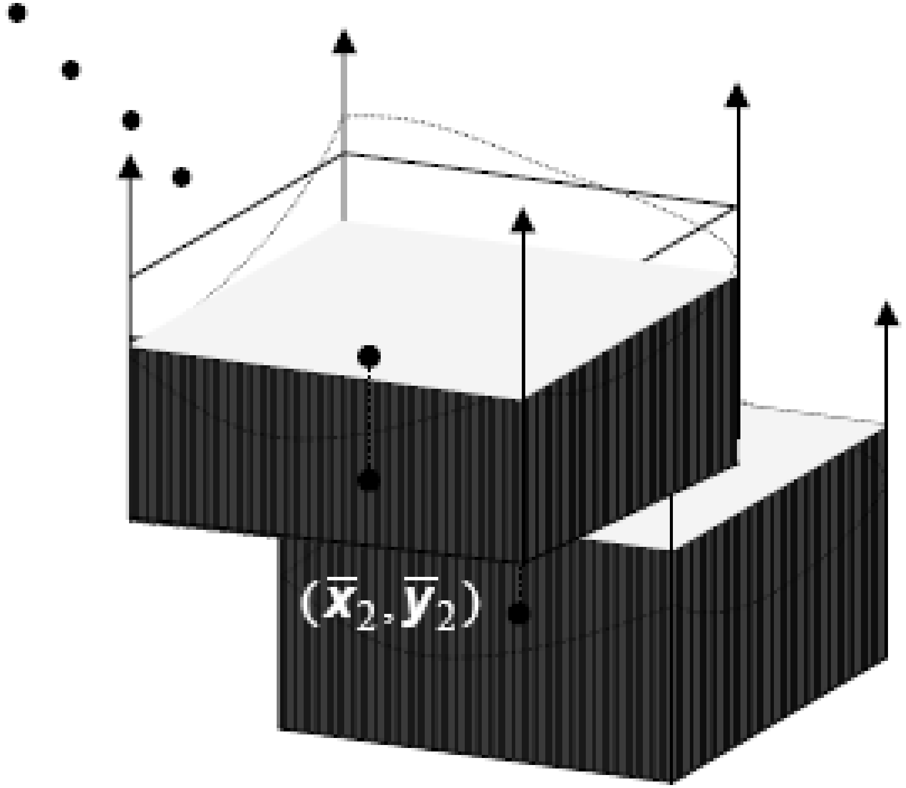


Code: mc_int_ex.m

MC Integration: 2D

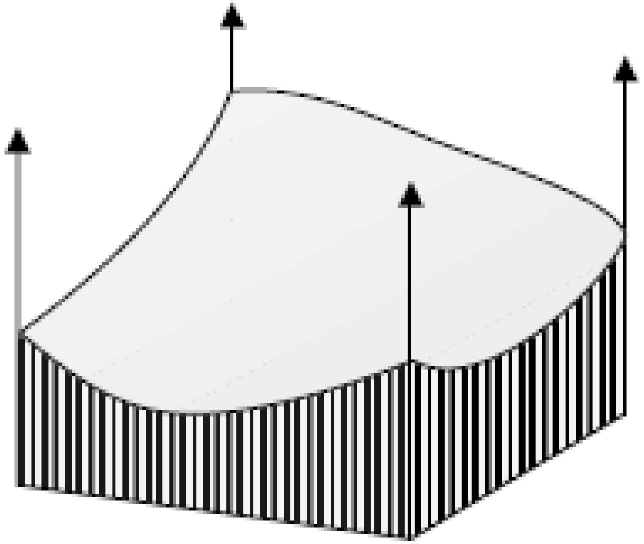
- Secondary estimator:

$$I_{\text{sec}} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(\bar{x}_i, \bar{y}_i)}{p(\bar{x}_i, \bar{y}_i)}$$



Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature



$$I = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

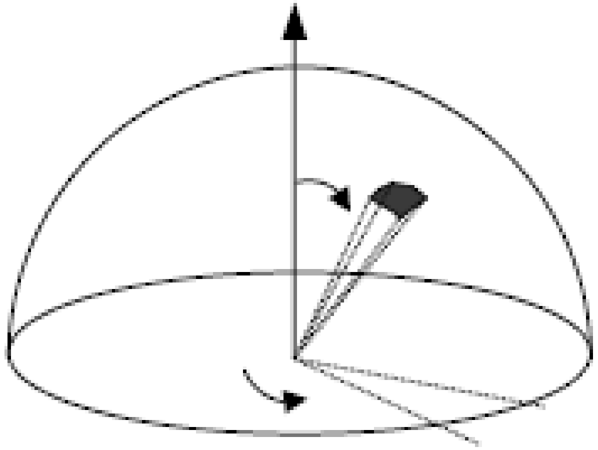
$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i, y_i)}{p(x_i, y_i)}$$

Advantages of MC

- **Convergence rate of** $O(\frac{1}{\sqrt{N}})$
- **Simple**
  - **Sampling**
  - **Point evaluation**
- **General**
  - **Works for high dimensions**
  - **Deals with discontinuities, crazy functions, etc.**

MC Integration - 2D example

- Integration over hemisphere:



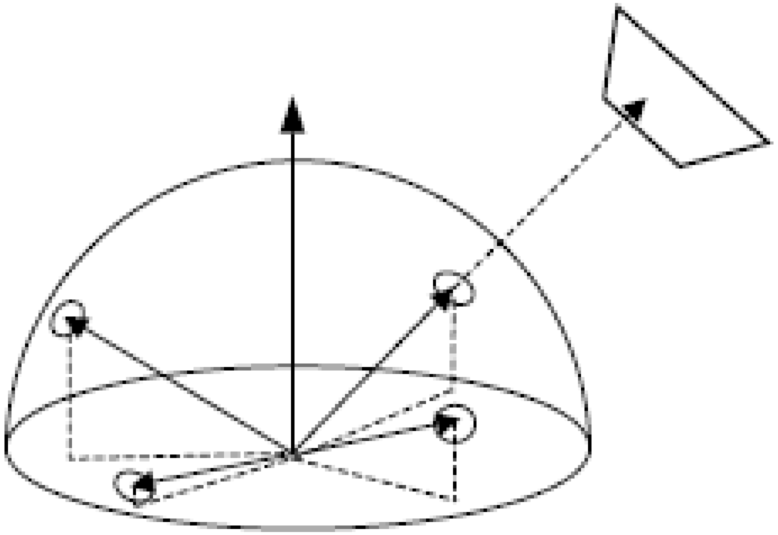
$$I = \int_{\Omega} f(\Theta) d\omega_{\Theta}$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi/2} f(\varphi, \theta) \sin\theta \, d\theta \, d\varphi$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\varphi_i, \theta_i) \sin\theta}{p(\varphi_i, \theta_i)}$$

Hemisphere Integration example

Irradiance due to light source:



$$I = \int_{\Omega} L_{source} \cos\theta \, d\omega_{\Theta}$$

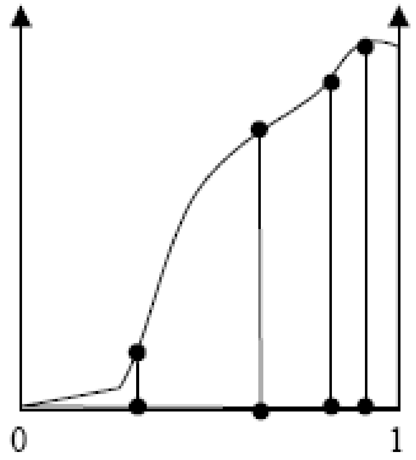
$$= \int_0^{2\pi} \int_0^{\pi/2} L_{source} \cos\theta \sin\theta \, d\theta \, d\varphi$$

$$p(\omega_i) = \frac{\cos\theta \sin\theta}{\pi}$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{L_{source}(\omega_i) \cos\theta \sin\theta}{p(\omega_i)} = \frac{\pi}{N} \sum_{i=1}^{N} L_{source}(\omega_i)$$

Importance Sampling

- Take more samples in important regions, where the function is large



From kavita's slides

- Sampling according to pdf (Ch. 14.4 Generating Samples)
  - Inverse cumulative distribution function
  - Rejection sampling

Class Objectives (Ch. 14) were:

- Sampling approach for solving the rendering equation
  - Monte Carlo integration
  - Estimator and its variance

Next Time…

- **Monte Carlo ray tracing**

Homework

- Go over the next lecture slides before the class
- Watch two videos and submit your summaries every Tue. class
  - Just one paragraph for each summary

Example:

Title: XXX XXXX XXXX

Abstract: this video is about accelerating the performance of ray tracing. To achieve its goal, they design a new technique for reordering rays, since by doing so, they can improve the ray coherence and thus improve the overall performance.

Any Questions?

- Submit three times before the mid-term exam
- Come up with one question on what we have discussed in the class and submit:
  - 1 for already answered questions
  - 2 for questions that have some thoughts or surprise me