CS380: Computer Graphics
3D Transformation

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Course URL:
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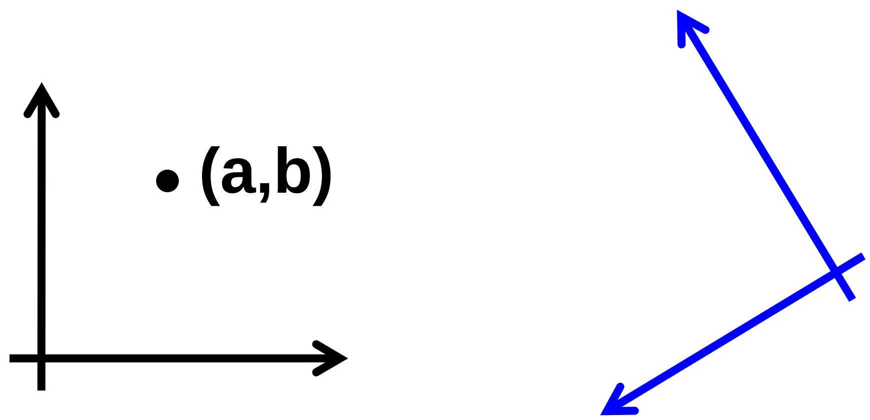
Class Objectives

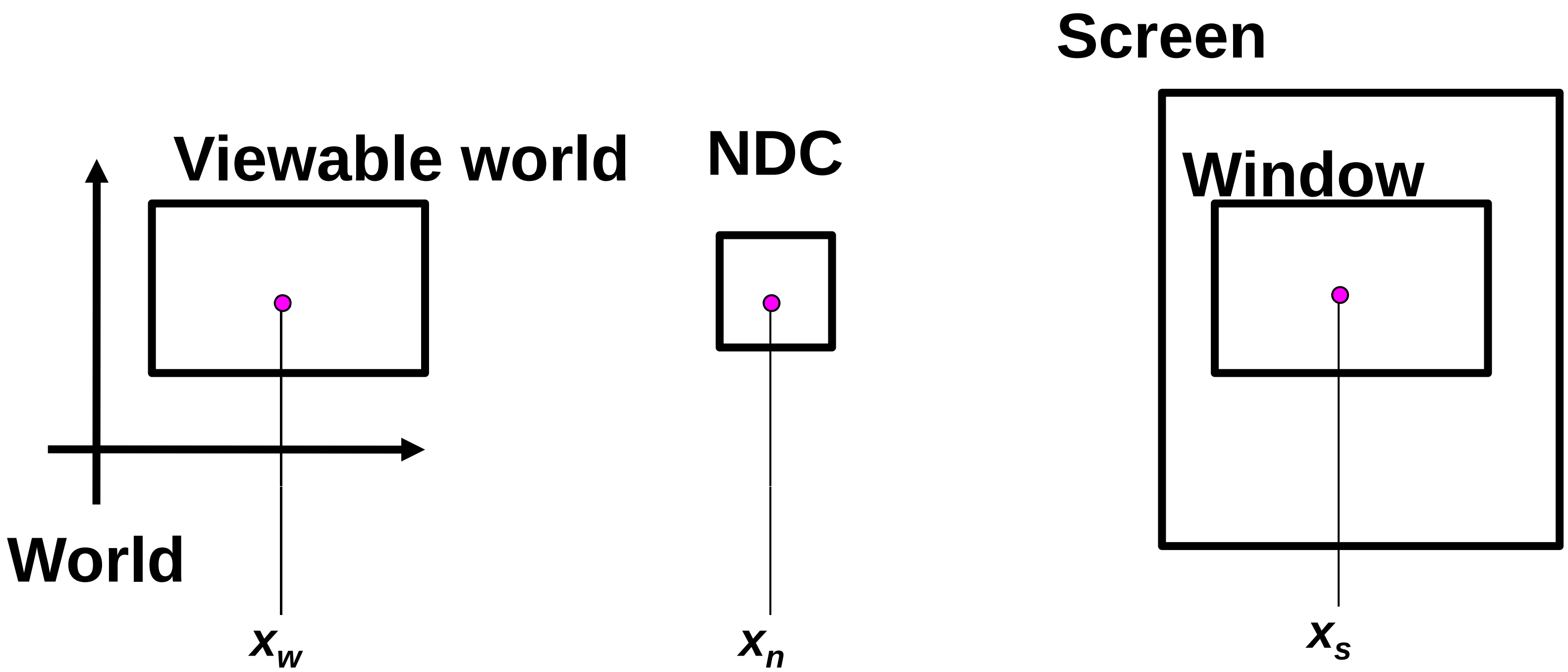
- **Understand the diff. between points and vectors**
- **Understand the frame**
- **Represent transformations in local and global frames**

A Question?

- **Suppose you have 2 frames and you know the coordinates of a point relative in one frame**
  - **How would you compute the coordinate of your point relative to the other frame?**
  - **(Generalized question to the mapping problem that we went over in the class)**



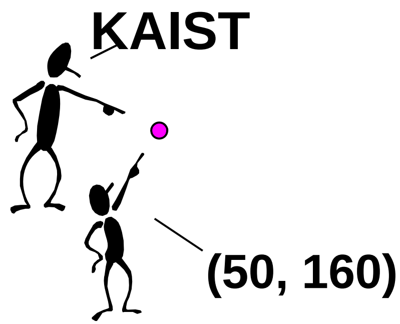
Revisit: Mapping from World to Screen

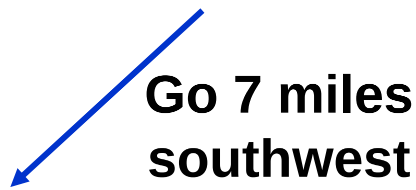


Geometry

- **A part of mathematics concerned with questions of size, shape, and relative positions of figures**

- **Coordinates are used to represent points and vectors**
  - **We will learn that they are just a naming scheme**
  - **The same point can be described by different coordinates**
  - **Both vectors and points expressed by coordinates, but they are very different**





Scalar Fields

- A scalar field **S is a set** on which addition (+) and multiplication (·) are defined and following conditions hold:
  - S is closed for addition and multiplication

$$\forall a,b \in S \quad a+b \in S \quad a \cdot b \in S$$

  - These operators commute, associate, and distribute

$$\forall a,b,c \in S$$

$$a+b=b+a \quad a \cdot b = b \cdot a$$

$$a+(b+c)=(a+b)+c \quad a \cdot (b \cdot c)=(a \cdot b) \cdot c$$

$$a \cdot (b+c)=a \cdot b + a \cdot c$$

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Scalar Fields – cont'd

- **A scalar field S is a set on which addition (+) and multiplication (·) are defined and following conditions hold:**
  - **Both operators have a unique identity element**

    $$a + 0 = a, \quad a \cdot 1 = a$$

  - **Each element has a unique inverse under both operators**

    $$a + (-a) = 0, \quad a \cdot a^{-1} = 1$$

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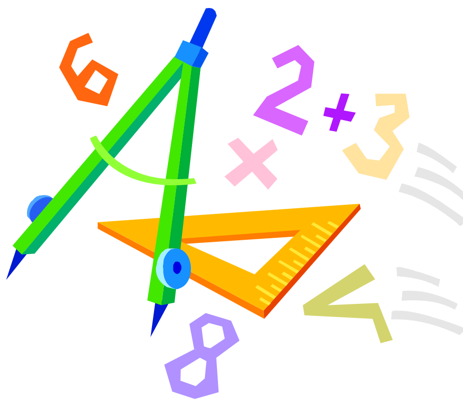
Examples of Scalar Fields

- **Real numbers**
- **Complex numbers (given the standard definitions for addition and multiplication)**
- **Rational numbers**
- **Notation: we will represent scalars by lower case letters**



a, b, c, … are scalar variables

Vector Spaces

- **A vector (or linear) space V over a scalar field S consists of a set on which the following two operators are defined and the following conditions hold:**
- **Two operators for vectors:**
  - **Vector-vector addition**

    $$\forall \vec{u}, \vec{v} \in V \quad \vec{u} + \vec{v} \in V$$

  - **Scalar-vector multiplication**

    $$\forall \vec{u} \in V, \forall a \in S \quad a\vec{u} \in V$$

- **Notation:**
  - **Vector** $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}^t$

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Vector Spaces

- **Vector-vector addition**
  - **Commutes and associates**

    $$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

  - **An additive identity and an additive inverse for each vector**

    $$\vec{u} + \vec{0} = \vec{u} \quad \vec{u} + (-\vec{u}) = \vec{0}$$

- **Scalar-vector multiplication distributes**

  $$(a + b)\vec{u} = a\vec{u} + b\vec{u} \quad a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$

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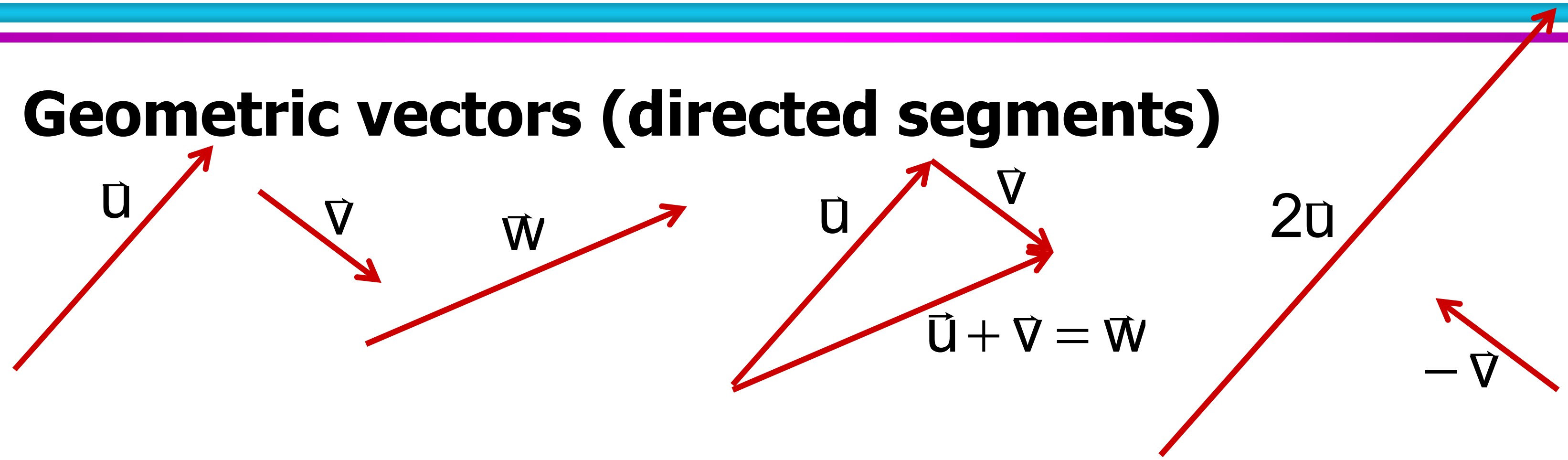
Example Vector Spaces

- **Geometric vectors (directed segments)**



- **N-tuples of scalars**

$$\vec{u}=(1,3,7)^t \qquad \vec{u}+\vec{v}=(3,5,4)^t=\vec{w}$$

$$\vec{v}=(2,2,-3)^t \qquad 2\vec{u}=(2,6,14)^t$$

$$\vec{w}=(3,5,4)^t \qquad -\vec{v}=(-2,-2,3)^t$$

- **We can use N-tuples to represent vectors**

Basis Vectors

- **A vector basis is a subset of vectors from V that can be used to generate any other element in V, using just additions and scalar multiplications**
- **A basis set, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, is linearly dependent if:**

$$\exists a_1, a_2, \ldots, a_n \neq 0 \quad \text{such that} \quad \sum_{i=0}^{n} a_i \vec{v}_i = 0$$

- **Otherwise, the basis set is linearly independent**
  - **A linearly independent basis set with *i* elements is said to *span* an *i-dimensional* vector space**

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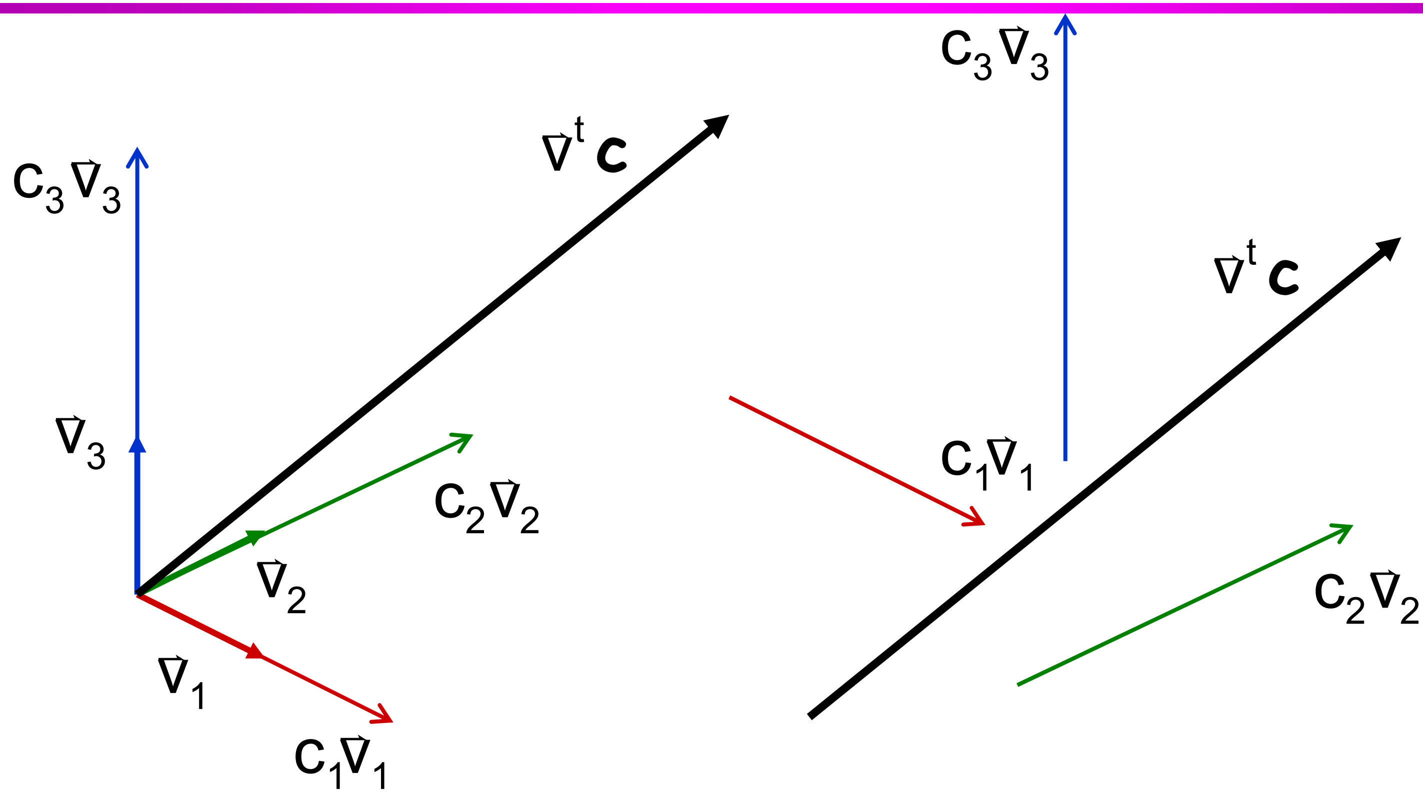
Vector Coordinates

- **A linearly independent basis set can be used to uniquely name or address a vector**
  - **This is the done by assigning the vector coordinates as follows:**

$$\vec{X} = \sum_{i=1}^{3} c_i \vec{V}_i = \begin{bmatrix} \vec{V}_1 & \vec{V}_2 & \vec{V}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{V}^t \mathbf{c}$$

- **Note: we'll use bold letters to indicate tuples of scalars that are interpreted as coordinates**
- **Our vectors are still abstract entities**
  - **So how do we interpret the equation above?**

Interpreting Vector Coordinates



Valid Interpretation **Equally Valid Interpretation**

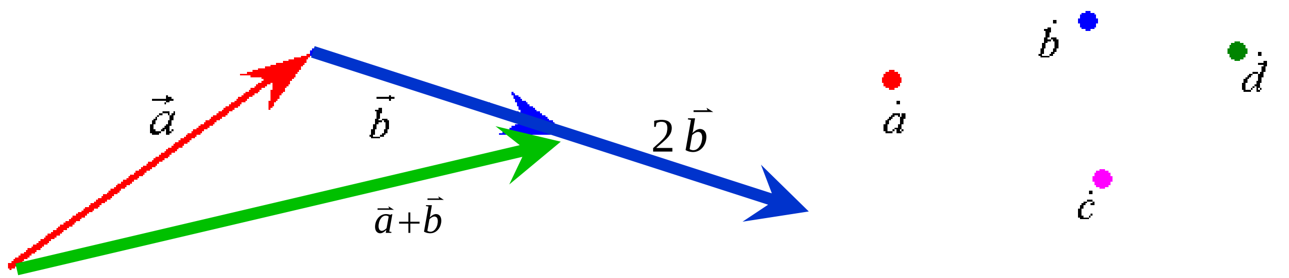
Remember, vectors don’t have any notion of position

Points

- **Conceptually, points and vectors are very different**
  - **A point** $\dot{p}$ **is a place in space**
  - **A vector** $\vec{v}$ **describes a direction independent of position (pay attentions notations)**

How Vectors and Points Differ

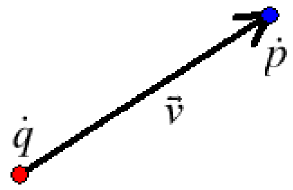
- The operations of addition and multiplication by a scalar are well defined for vectors
  - Addition of 2 vectors expresses the concatenation of 2 “motions”
  - Multiplying a vector by some factor scales the motion
- These operations does not make sense for points



Making Sense of Points

- Some operations **do make sense** for points
  - Compute a vector that describes the motion from one point to another:

$$\dot{p} - \dot{q} = \vec{v}$$



  - Find a new point that is some vector away from a given point:

$$\dot{q} + \vec{v} = \dot{p}$$

A Basis for Points

- **Key distinction between vectors and points: points are *absolute,* vectors are *relative***
- **Vector space is completely defined by a set of basis vectors**
- **The space that points live in requires the specification of an absolute origin**

$$\dot{p} = \dot{o} + \sum_i \vec{v}_i c_i = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

Notice how 4 scalars (one of which is 1) are required to identify a 3D point

Frames

- **Points live in *Affine* spaces**
- **Affine-basis-sets are called *frames***

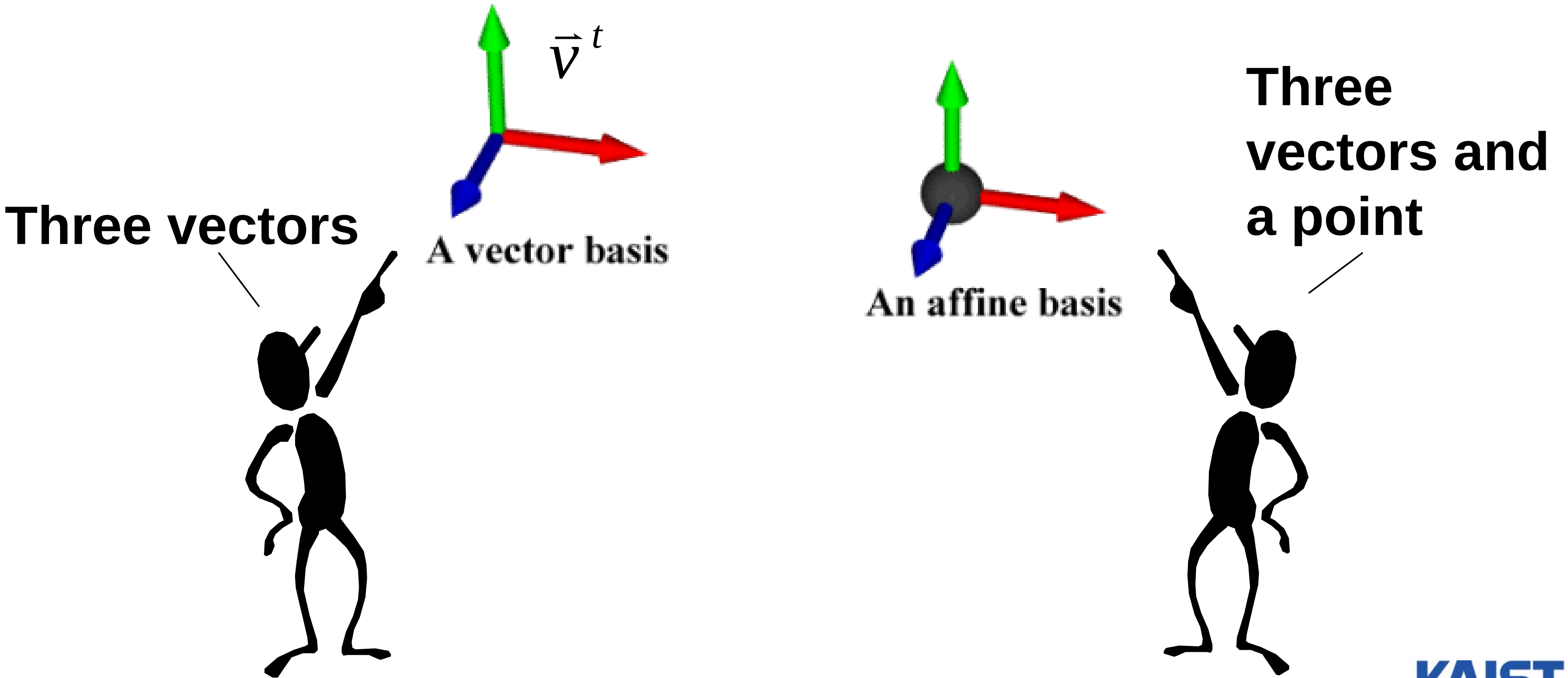
$$\dot{f}^{t} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix}$$

- **Frames can describe vectors as well as points**

$$\dot{p} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$$

Pictures of Frames

- **Graphically, we will distinguish between vector bases and affine bases (frames) using the following convention**



A Consistent Model

- **Behavior of affine frame coordinates is completely consistent with our intuition**
  - **Subtracting two points yields a vector**
  - **Adding a vector to a point produces a point**
  - **If you multiply a vector by a scalar you still get a vector**
  - **Scaling points gives a nonsense 4^{th} coordinate element in most cases**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + v_1 \\ a_2 + v_2 \\ a_3 + v_3 \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- **Notice why we introduce homogeneous coordinates, based on simple logical arguments**
  - **Remember that coordinates are not geometric; they are just scales for basis elements**
  - **Thus, you should not be bothered by the fact that our coordinates suddenly have 4 numbers**

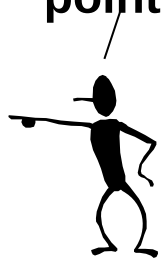
- **3D homogeneous coordinates refer to an affine frame with its 3 basis vectors and origin point**
  - **4 coordinates make sense in this aspect**
  - **4th coordinate can have one of two values, [0,1], indicating if whether the coordinates name a vector or a point**

Affine Combinations

- **There are certain situations where it makes sense to scale and add points**
  - **Suppose you have two points, one scaled by a_1 and the other scaled by a_2**
  - **If we restrict the sum of these alphas, $a_1 + a_2 = 1$, we can assure that the result will have *1* as it's 4th coordinate value**

$$\alpha_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 b_1 \\ \alpha_1 a_2 + \alpha_2 b_2 \\ \alpha_1 a_3 + \alpha_2 b_3 \\ \alpha_1 + \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 a_1 + \alpha_2 b_1 \\ \alpha_1 a_2 + \alpha_2 b_2 \\ \alpha_1 a_3 + \alpha_2 b_3 \\ 1 \end{bmatrix}$$

But, is it a point?



Affine Combinations

- **Can be thought of as a constrained-scaled addition**
  - **Defines all points that share the line connecting our two initial points**

- **Can be extended to 3, 4, or any number of points (e.g., barycentric coordinates)**

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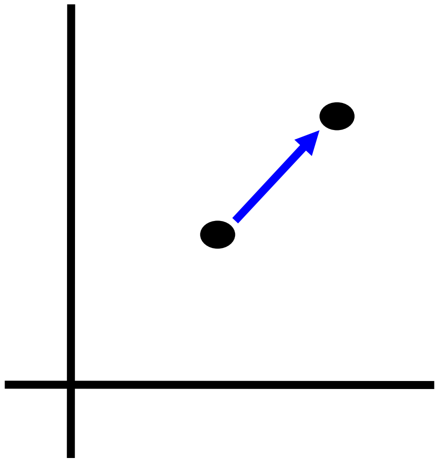
Affine Transformations

- **We can apply transformations to points using matrix**
  - **Need to use 4 by 4 matrices since our basis set has four components**
  - **Also, limit ourselves to transforms that preserve the integrity of our points and vectors; point to point, vector to vector**

$$\dot{p} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \Rightarrow \dot{p}' = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dot{o} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

  - **This subset of matrices is called the *affine* subset**

An Example



Composing Transformations

- **Represent a series of transformations**
  - **E.g., want to translate with T and, then, rotate with R**

- **Then, the series is represented by:**

$$\dot{p} = \dot{w}^t c \Rightarrow \dot{p}' = \dot{w}^t RTc = \dot{w}^t (R(Tc)) = \dot{w}^t (Rc') = \dot{w}^t c''$$

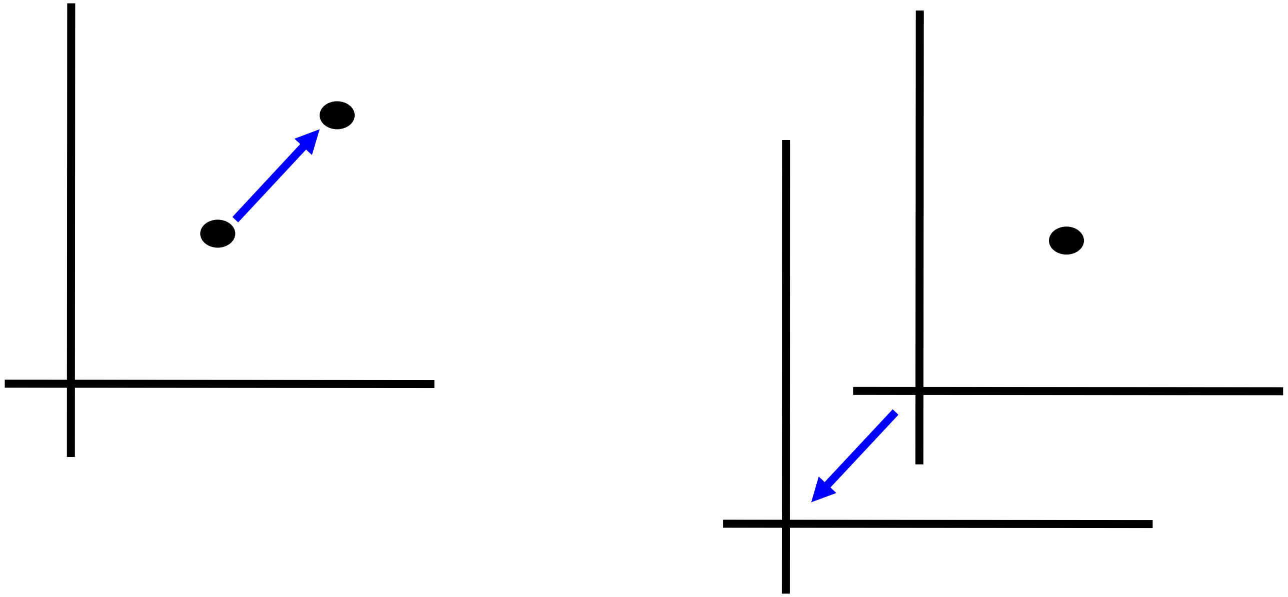
  - **Each step in the process can be considered as a change of coordinates**

- **Alternatively, we could have considered the same sequence of operations as:**

$$\dot{p} = \dot{w}^t c \Rightarrow \dot{p}' = \dot{w}^t RTc = ((\dot{w}^t R)T)c = (\dot{m}^t T)c = \dot{e}^t c$$

, where each step is considered as a change of basis

An Example



- **These are alternate interpretations of the same transformations**
  - **The left and right sequence are considered as a transformation about a *global frame and local* frames**

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Same Point in Different Frames

- **Suppose you have 2 frames and you know the coordinates of a point relative in one frame**
  - **How would you compute the coordinate of your point relative to the other frame?**

$$\vec{p} = \vec{w}^t \mathbf{c} = \vec{z}^t \ ?$$

  - **Suppose that my two frames are related by the transform *S* as shown below:**

$$\vec{z}^t = \vec{w}^t \mathbf{S} \quad \text{and} \quad \vec{w}^t = \vec{z}^t \mathbf{S}^{-1}$$

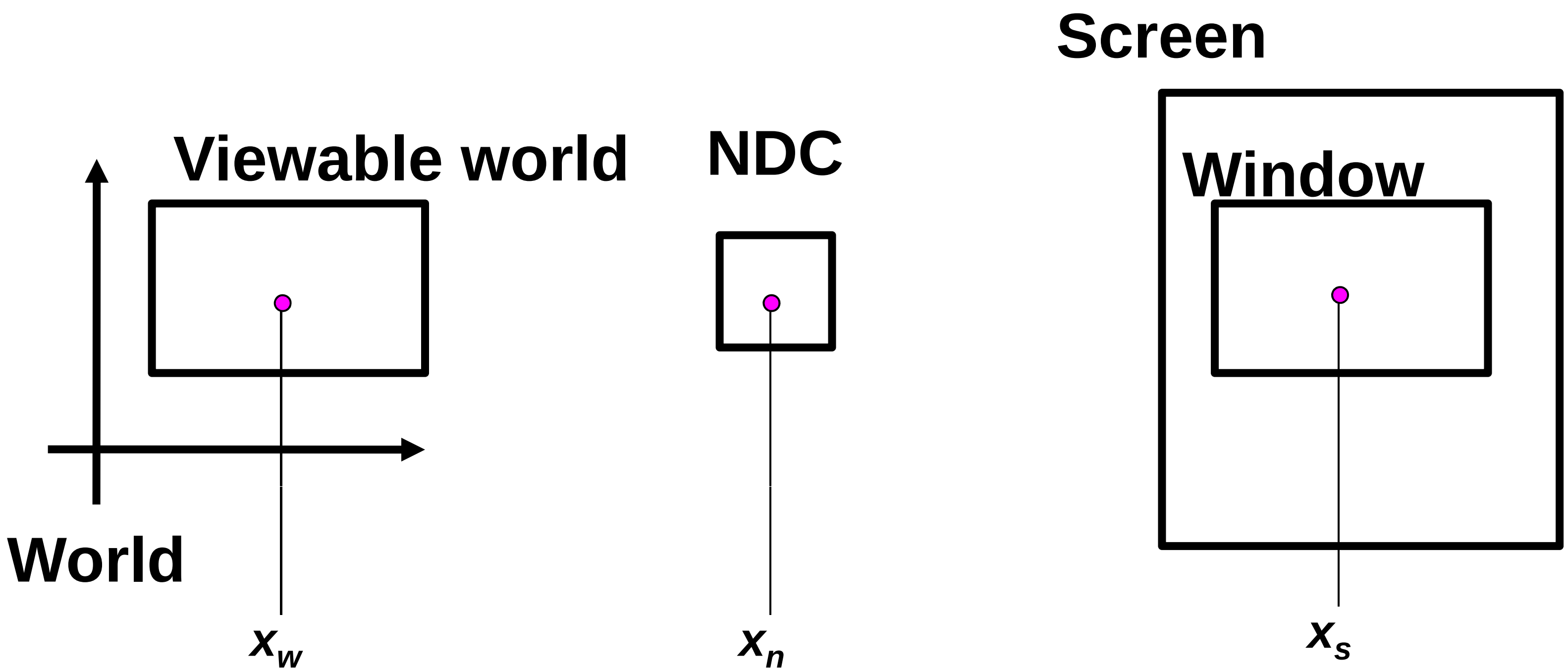
  - **Then, the coordinate for the point in second frame is simply:**

$$\vec{p} = \vec{w}^t \mathbf{c} = \vec{z}^t \mathbf{S}^{-1} \mathbf{c} = \vec{z}^t (\mathbf{S}^{-1} \mathbf{c}) = \vec{z}^t \mathbf{d}$$

Substitute for the frame

Reorganize & reinterpret

Revisit: Mapping from World to Screen



Class Objectives were:

- **Understand the diff. between points and vectors**
- **Understand the frame**
- **Represent transformations in local and global frames**

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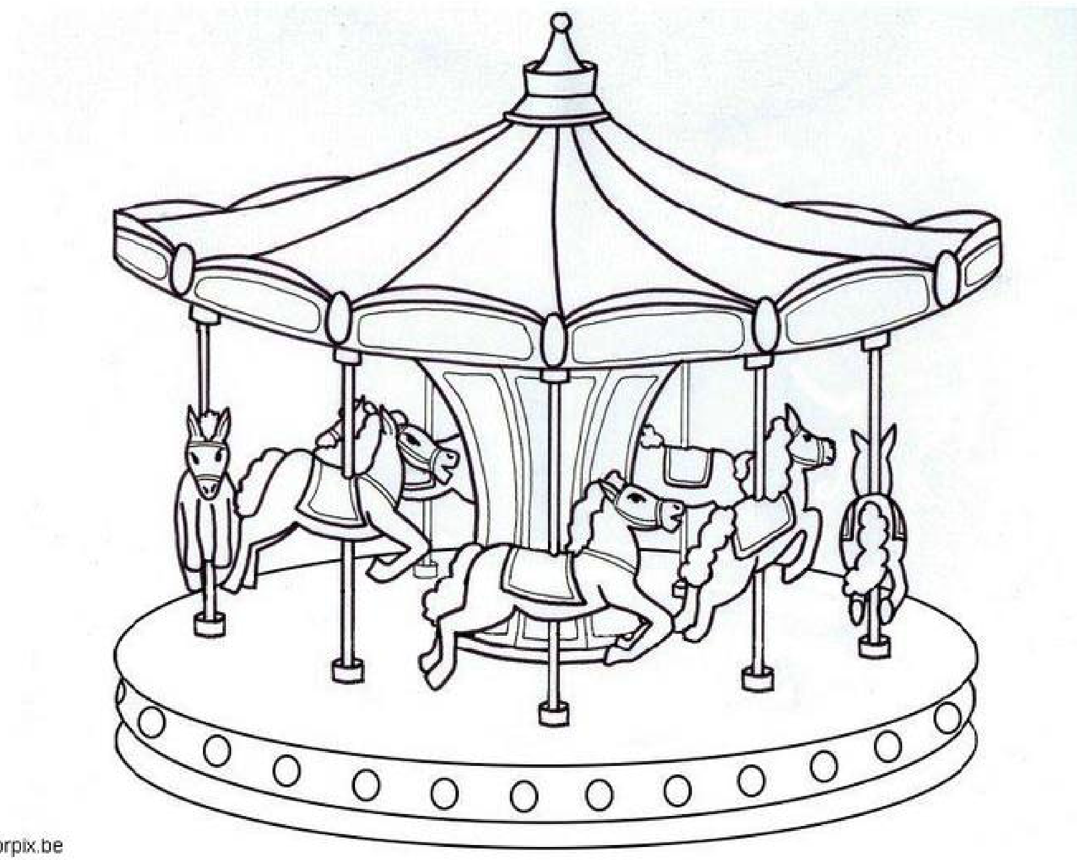
Quiz Assignment



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Next Time

- **Modeling and viewing transformations**